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Addendum

A Stochastic Particle System Modeling the Carleman Equation[1]

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This addendum concerns the statement after (3.19), which says that the hierarchy of equations (3.11) has a unique solution [in (3.11) there should be a minus sign in front of $\sigma\Delta$].

Such a statement is easily proven when one has an *a priori* bound on the j-body correlation functions of the form c^j, with some fixed c which does not depend on j. Our *a priori* bound, however, grows like e^{cj^2} and the proof of uniqueness becomes much more delicate: uniqueness might not even be true, in general, in the class of correlation functions bounded only by e^{cj^2}. Our proof, as we are going to see, exploits in an essential way the presence of the heat kernel, $\sigma \neq 0$, in the same way used to prove the *a priori* bound.

We write in integral form the hierarchy of equations (3.11) for the correlation functions h_j^σ and we get

$$h_j^\sigma(\cdot, t) = V_{j,t} f_{j,0} + \int_0^t ds\, V_{j,t-s} C_{j,j+1} h_{j+1}^\sigma(\cdot, s) \tag{1}$$

where $f_{j,0}$ are the correlation functions at time 0, namely

$$f_{j,0} = \prod_{k=1}^{j} f(x_k, v_k, 0) \tag{2}$$

$f(x, v, 0)$ being the initial datum for the Carleman equation, assumed to be

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a normalized, nonnegative function in $\mathbb{C}^0([0,1]^2)$. The collision operator $C_{j,j+1}$ is given by

$$C_{j,j+1}=\sum_{i=1}^{j}\sum_{b_i=\pm} C_{j,j+1}^{i,b_i} \tag{3a}$$

$$\begin{aligned} C_{j,j+1}^{i,\pm}\, & g_{j+1}(x_1,v_1,\ldots,x_j,v_j) \\ & = \mp g_{j+1}(x_1,v_1,\ldots,x_i,\pm v_i,\ldots,x_j,v_j,x_i,\pm v_i) \end{aligned} \tag{3b}$$

Iterating (1) n times, we get

$$\begin{aligned} h_j^\sigma(\cdot,t) &= V_{j,t} f_{j,0} \\ &+ \sum_{m=1}^{n-1}\int_0^t ds_1\cdots\int_0^{s_{m-1}} ds_m \sum_{i_1,\ldots,i_m}\sum_{b_1,\ldots,b_m} V_{j,t-s_1} C_{j,j+1}^{i_1,b_1}\cdots \\ &\times V_{j+m-1,s_{m-1}-s_m} C_{j+m-1,j+m}^{i_m,b_m} V_{j+m,s_m} f_{j+m,0} \\ &+ \int_0^t ds_1\cdots\int_0^{s_{n-1}} ds_n \sum_{i_1,\ldots,i_n}\sum_{b_1,\ldots,b_n} V_{j,t-s_1} C_{j,j+1}^{i_1,b_1}\cdots \\ &\times V_{j+n-1,s_{n-1}-s_n} C_{j+n-1,j+n}^{i_n,b_n} h_{j+n}^\sigma(\cdot,s_n) \end{aligned} \tag{4}$$

We bound the first two terms on the right-hand side of (4) by

$$\sum_{m=0}^{n-1}\frac{(j+m)!}{j!\,m!}2^m t^m c^{j+m} \tag{5}$$

where $c\equiv \max f(x,v,0)$. As we shall see at the end of this addendum, we can also prove a bound which is independent of the sup-norm of the initial datum.

For t small enough, precisely for $2tc<1$, (5) is bounded uniformly on n. The problem is therefore to control the remainder in (4). Let us fix the values $n\geqslant 2$, $s_1,\ldots,s_n$, $i_1,\ldots,i_n$, and $b_1,\ldots,b_n$. Call

$$A\equiv \left|C_{j+n-2,j+n-1}^{i_{n-1},b_{n-1}} V_{j+n-1,s_{n-1}-s_n} C_{j+n-1,j+n}^{i_n,b_n} h_{j+n}^\sigma(\cdot,s_n)\right| \tag{6}$$

Assume first that the label $j+n-1\neq i_n$; then, by integrating over the $(j+n-1)$th particle and using (3.3b) [see (3.2) for notation] and (3.8), we get

$$A\leqslant c_\sigma(s_{n-1}-s_n)^{-1/2}\left|V_{j+n-2,s_{n-1}-s_n} C_{j+n-2,j+n-1}^{i_n,b_n} h_{j+n-1}^\sigma(\cdot,s_n)\right| \tag{7}$$

If, on the other hand, $i_n=j+n-1$, we can use the symmetry of A under the exchange of $j+n-1$ and i_{n-1} due to the fact that the particles with labels $j+n-1$ and i_{n-1} are in the same state at time s_{n-1}.

We can now use the semigroup property and iterate the procedure. We therefore have that the last integral in (4) is bounded, in absolute value, by

$$\frac{(j+n)!}{j!} 2^n c_\sigma^{n-1} \|h_{j+1}^\sigma\|_\infty \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n (s_1 - s_n)^{-1/2} \cdots (s_{n-1} - s_n)^{-1/2} \quad (8)$$

where $\|h_{j+1}^\sigma\|_\infty$ is the sup over space and over times $s \leqslant (2c)^{-1}$, c being as in (5): we consider for the moment only times t which are smaller than $(2c)^{-1}$. The expression in (8) is equal to

$$\frac{(j+n)!}{j!} 2^n c_\sigma^n \|h_{j+1}^\sigma\|_\infty \frac{2^{n+1}}{(n-1)!\,(n+1)} t^{(n+1)/2} \quad (9)$$

For $4c_\sigma t^{1/2} < 1$ this term vanishes when $n \to \infty$. Hence, for $t < \min\{(4c_\sigma)^{-2}, (2c)^{-1}\}$, [cf. (5)] $h_j^\sigma(\cdot, t)$ is given by the limit as $n \to \infty$ of the first two terms in (4). This proves that in the same time interval

$$h_j^\sigma(\cdot, t) = \prod_{i=1}^{j} f(\cdot, t) \quad (10)$$

where f solves (2.17). By (3.18) we can start again and reach times twice as large as before. By iteration we then prove that (10) extends to all times.

A final remark: the same argument used to control the last integral in (4) allows one to prove a bound for the first sum in (4) which is independent of $\|f(\cdot, 0)\|_\infty$.